A UNIFIED CLASS OF INVERSE FILTER CRITERIA USING TWO CUMULANTS FOR BLIND DECONVOLUTION AND EQUALIZATION

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ABSTRACT

Cumulant (higher-order statistics) based inverse filter criteria maximizing $J_{r,m} = |C_m|^r/|C_r|^m$, where $m \neq r$ and C_m (C_r) denotes the mth-order (rth-order) cumulant of the inverse filter output, have been proposed for blind deconvolution and equalization with only non-Gaussian output measurements of an unknown linear time-invariant (LTI) system. This paper shows that the maximum of $J_{r,m}$, associated with the true inverse filter of the unknown LTI system, exists only for r to be even and $m > r$, otherwise, $J_{r,m}$ is unbounded. The admissible values for $(r,m) = (2s, l+s)$ where $l > s \geq 1$ include $(2,3)$, $(2,4)$ and $(4,6)$ proposed by Tugnait, Wiggins, Shalvi and Weinstein in addition to more new ones such as $(2,5)$, $(2,6)$ and $(4,5)$. Some simulation results for the inverse filter criteria $J_{r,m}$ with the proposed admissible values of (r,m) are then provided. Finally, we draw some conclusions.

1. INTRODUCTION

Blind deconvolution as well as equalization is a quite known statistical signal processing problem to estimate the desired signal $u(n)$ with a given set of measurements $x(n)$, $n = 0, 1, \ldots, N-1$ based on the following convolutional model

$$
\begin{aligned}
x(n) &= u(n) * h(n) + w(n) \\
&= \sum_{i=-\infty}^{\infty} h(i)u(n-i) + w(n)
\end{aligned}
\tag{1}
$$

where $w(n)$ is measurement noise and $h(n)$ is an unknown linear time-invariant (LTI) system which corresponds to such as the source wavelet in seismic deconvolution, the channel impulse response in channel equalization and the vocal-tract filter in speech processing. A major conventional approach to this problem is the correlation (second-order statistics) based predictive deconvolution. The deconvolved signal is obtained by processing $x(n)$ with a minimum-phase linear prediction error (LPE) filter, which corresponds to an estimate for the inverse filter of $h(n)$ except for phase distortion because $h(n)$ may not be minimum-phase in practice. Recently, higher-order (≥ 3) statistics (HOS) [1,2], known as cumulants, have been considered in various signal processing areas where $x(n)$ is non-Gaussian

and measurement noise $w(n)$ is Gaussian with unknown statistics, partly because cumulants of $x(n)$ can be used to extract not only the amplitude information but also phase information of $h(n)$ and partly because higher-order cumulants of Gaussian noise $w(n)$ are zero.

Cumulant based inverse filter criteria [3-7] have been considered for the estimation of the inverse filter $h_I(n)$ of the unknown LTI system $h(n)$. Assume that $v(n)$ is an estimate for $h_I(n)$ and $e(n)$ is the output signal of $v(n)$ in response to $x(n)$, i.e.,

$$
e(n) = x(n) * v(n)
\tag{2}
$$

and C_m denotes the mth-order cumulant of $e(n)$. A class of cumulant based inverse filter criteria has the following form [3-5]:

$$
J_{r,m}(v(n)) = \frac{|C_m|^r}{|C_r|^m}
\tag{3}
$$

where $m \neq r$, $m \geq 2$ and $r \geq 2$. Wiggins [3] proposed an inverse filter criterion by maximizing $J = E[e^4(n)]/(E[e^2(n)])^2$, which is related to $J_{r,m}$ by $|J - 3|^2 = J_{2,4}$. Shalvi and Weinstein [4] proposed an inverse filter criterion by maximizing $|C_4|$ subject to the constraint $E[e^2(n)] = E[u^2(n)]$. Tugnait [5] also proposed inverse filter criteria by maximizing $J_{2,3}$ or $J_{2,4}$, or $J_{4,6}$. However, for other choices of r and m, it is still unknown whether maximizing $J_{r,m}$ can lead to the true inverse filter of $h(n)$. Chi and Kung [6] estimated the inverse filter by maximizing a single cumulant $|C_m|$ for $m \geq 3$ when $h(n)$ is an allpass system.

In this paper, we show that maximizing the objective function $J_{r,m}$ given by (3), which only uses two cumulants of $e(n)$, is applicable only for some certain choices of r and m which lead to more new inverse filter criteria in addition to the aforementioned existing inverse filter criteria as special cases of the admissible r and m.

2. ADMISSIBLE CUMULANT ORDERS FOR THE INVERSE FILTER CRITERIA $J_{r,m}$

First of all, let us make the following assumptions for measurements $x(n)$ modeled by (1).

(A1) The unknown LTI system $h(n)$ is causal stable with either minimum phase or nonminimum phase and a stable inverse filter $h_I(n)$ of $h(n)$ exists.

(A2) The driving input $u(n)$ is real, zero-mean, stationary, independent identically distributed (i.i.d.) non-Gaussian with variance σ_u^2 and mth-order cumulant γ_m where $m \geq 3$.

(A3) Measurement noise $w(n)$ is Gaussian with unknown statistics.

(A4) The input $u(n)$ and the noise $w(n)$ are statistically independent.

The admissible cumulant orders for the inverse filter criteria given by (3) are described in the following theorem.

Theorem 1. Assume that $x(n)$ is the noisy signal generated from the model given by (1) under the previous assumptions **(A1)** through **(A4)**. Then the following two statements are true:

(S1) $J_{r,m}(v(n))$ is unbounded except the case that $r = 2s$ (i.e., r is even), $m = l+s > r$ where $l > s \geq 1$. Moreover,

$$max\{J_{2s,l+s}(v(n))\} = \frac{\mid \gamma_{l+s} \mid^{2s}}{\mid \gamma_{2s} \mid^{l+s}} \qquad (4)$$

(S2) The optimum $\hat{v}(n)$ associated with $J_{2s,l+s}(\hat{v}(n))$ $= max\{J_{2s,l+s}(v(n))\}$ where $l > s \geq 1$ satisfies

$$\hat{v}(n) * h(n) = \alpha\delta(n - \tau) \qquad (5)$$

where $\alpha \neq 0$ is a scale factor and τ is an unknown integer, for the two cases that $s = 1$ with $SNR = \infty$ and that $s > 1$ with finite SNR.

Two remarks for the inverse filter criteria $J_{2s,l+s}$ are summarized as follows:

(R1) For $(s, l) = (1,2)$, $(s,l)=(1,3)$, and $(s,l)=(2,4)$, $J_{2s,l+s}$ reduce to Tugnait's inverse filter criteria $J_{2,3}$, $J_{2,4}$ and $J_{4,6}$, respectively. For any other choices of (s, l), $J_{2s,l+s}$ such as $J_{2,5}$, $J_{2,6}$ and $J_{4,5}$ are new.

(R2) In practice, the two cumulants C_{2s} and C_{l+s} required by $J_{2s,l+s}$ must be replaced by the corresponding sample cumulants $\widehat{C}_{2s}$ and $\widehat{C}_{l+s}$ which are well-known consistent estimates for C_{2s} and C_{l+s} [2], respectively. Therefore, the optimum estimate $\hat{v}(n)$ is also a consistent estimate for the inverse filter $h_I(n)$ except for a scale factor and an unknown time delay.

Next, let us present how we find the optimum inverse filter $\hat{v}(n)$ associated with the inverse filter criteria $J_{2s,l+s}$ with finite data set $\{x(0), x(1), \ldots, x(N-1)\}$. The inverse filter $v(n)$ is assumed to be a causal FIR filter of order equal to L. Then the inverse filter output $e(n)$ given by (2) can be expressed as

$$e(n) = \mathbf{v}^T\mathbf{x}_n \qquad (6)$$

where $\mathbf{v}$ and $\mathbf{x}_n$ are $(L + 1) \times 1$ column vectors given by

$$\mathbf{v} = [v(0), \ v(1), \ \ldots, \ v(L)]^T \qquad (7)$$

and

$$\mathbf{x}_n = [x(n), \ x(n-1), \ \ldots, \ x(n-L)]^T, \qquad (8)$$

respectively. Note that the inverse filter criteria $J_{2s,l+s}$ where $l > s \geq 1$ are highly nonlinear functions of $\mathbf{v}$ since sample cumulants $\widehat{C}_{2s}$ and $\widehat{C}_{l+s}$ are nonlinear functions of $\mathbf{v}$. A gradient type numerical optimization algorithm is used to search for the optimum inverse filter estimate $\hat{\mathbf{v}}$. At the ith iteration $\hat{\mathbf{v}}_i$ is updated with

$$\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_{i-1} + \rho\mathbf{g}_{i-1} \qquad (9)$$

where ρ is a positive constant and $\mathbf{g}_{i-1}$ is the gradient of $J_{2s,l+s}$ with respect to $\mathbf{v}$ for $\mathbf{v} = \hat{\mathbf{v}}_{i-1}$. As other numerical optimization algorithms, an initial condition for $\hat{\mathbf{v}}_0$ is needed to initialize the above numerical optimization algorithm. For instance, a minimum-phase LPE filter can be used as the initial condition for $\hat{\mathbf{v}}_0$. Next, let us present some simulation results to justify that the inverse filter criteria $J_{2s,l+s}$ where $l > s \geq 1$ can be used for the estimation of the inverse filter of the unknown LTI system $h(n)$ and for deconvolution.

3. SIMULATION RESULTS

Two simulation examples are to be presented to support the proposed unified class of inverse filter criteria $J_{2s,l+s}$ where $l > s \geq 1$ presented in Theorem 1. The first example is a performance test to the inverse filter criterion $J_{2,3}$ proposed by Tugnait and the new $J_{2,5}$ (i.e., $s = 1$ and $l = 4$ in $J_{2s,l+s}$). The second example is seismic deconvolution using the new criterion $J_{2,6}$ (i.e., $s = 1$ and $l = 5$ in $J_{2s,l+s}$).

Example 1: (Performance test)

The driving input $u(n)$ used was a zero-mean, i.i.d. Exponential random sequence with variance $\sigma_u^2 = 1$, skewness $\gamma_3 = 2$, kurtosis $\gamma_4 = 6$ and fifth-order cumulant $\gamma_5 = 24$. The unknown LTI system $h(n)$ used was a nonminimum-phase second-order autoregressive moving average (ARMA) system with the transfer function (taken from [7]) given by

$$H(z) = \frac{1 - 2.7z^{-1} + 0.5z^{-2}}{1 + 0.1z^{-1} - 0.12z^{-2}} \qquad (10)$$

Synthetic noisy data $x(n)$ of length $N = 4096$ were generated for $SNR = 10dB$ and noise $w(n)$ being white Gaussian. The inverse filter $v(n)$ was assumed to be a causal FIR filter of order $L = 16$. An initial condition $\mathbf{v} = [0, \ldots, 0, 1, v_b(1), \ldots, v_b(8)]^T$ was used to initialize the associated gradient type optimization algorithm for finding the optimum $\hat{\mathbf{v}}$ where $\{1, v_b(1), \ldots, v_b(8)\}$ were the coefficients of an eighth-order LPE filter obtained by the well-known Burg's algorithm [8].

The simulation results over 30 independent runs associated with $J_{2,3}$ and $J_{2,5}$ (new criterion) are shown

in Figures 1(a) and 1(b), respectively. Note that each inverse filter estimate $\hat{\mathbf{v}}$ obtained from each independent run was normalized by $\|\hat{\mathbf{v}}\| = 1$ and the associated unknown time delay was artificially removed. From Figures 1(a) and 1(b), one can see that the estimated inverse filter is basically unbiased for both $J_{2,3}$ and $J_{2,5}$, but it has a smaller variance for $J_{2,3}$ than for $J_{2,5}$. The reason for this is simply that sample cumulant $\widehat{C}_5$ has larger variance than $\widehat{C}_3$ for finite data [2]. These simulation results justify that $J_{2s,l+s}$ where $l > s \geq 1$ can be used to estimate the inverse filter of an unknown nonminimum-phase LTI system.

With no doubt, when $J_{2,3}$ is sufficient in certain practical applications, $J_{r,m}$ for higher admissible r and m such as $J_{2,5}$ are redundant. Nevertheless, we would like to emphasize that when cumulants of measurements are small or close to zero for lower r and m such as $J_{2,3}$ and $J_{2,4}$, the inverse filter criteria $J_{r,m}$ only for higher admissible r and m such as $J_{2,5}$ can be considered for the estimation of inverse filter.

Example 2. (Seismic deconvolution)
Assume that $u(n)$ is a Bernoulli-Gaussian sequence (solid lines in Figures 2(b) and 2(c)) which is input to a third-order nonminimum-phase system (taken from [6]) with the following transfer function

$$H(z) = \frac{1 + 0.1z^{-1} - 3.2725z^{-2} + 1.41125z^{-3}}{1 - 1.9z^{-1} + 1.1525z^{-2} - 0.1625z^{-3}} \quad (11)$$

The synthetic noisy data $x(n)$ were generated for $N = 2048$, $SNR = 27dB$ and noise $w(n)$ being white Gaussian. Because $\gamma_3 = 0$ (skewness) and $\gamma_5 = 0$ but $\gamma_2 = \sigma_u^2 = 0.1$, $\gamma_4 = 0.27$ (kurtosis) and $\gamma_6 = 1.08$ for this case, the new inverse filter criterion $J_{2,6}$ was used to estimate the causal inverse filter $v(n)$ of order $L = 24$. The initial condition $\mathbf{v} = [0, \ldots, 0, 1, v_b(1), \ldots, v_b(12)]^T$, where $\{1, v_b(1), \ldots, v_b(12)\}$ were the coefficients of a twelfth-order LPE filter obtained by Burg's algorithm, was used to initialize the associated gradient type optimization algorithm for finding the optimum $\hat{\mathbf{v}}$.

The obtained inverse filter estimate (dash-dotted line) normalized by $\|\hat{\mathbf{v}}\| = 1$ is shown in Figure 2(a) together with the true noncausal stable inverse filter $h_I(n)$ (solid line) also normalized by $\sum_{n=-\infty}^{\infty} |h_I(n)|^2 = 1$ where the unknown time delay between $\hat{v}(n)$ and $h_I(n)$ was artificially removed. For comparison, a conventional minimum-phase LPE filter $v_b(n)$ of order equal to 24 was also obtained by Burg's algorithm, which is depicted by a dashed line in Figure 2(a). Note, from Figure 2(a), that $\hat{v}(n)$ is quite close to the true inverse filter $h_I(n)$ but very different from the LPE filter in waveshape. The data $x(n)$ were then processed by the LPE filter to obtain the predictive deconvolved signal $e_b(n)$ which is depicted by a dotted line in Figure 2(b) for $N = 0 \sim 511$ together with the true input sequence $u(n)$ depicted by a solid line. One can observe, from Figure 2(b), that in addition to a scale factor, each spike in $u(n)$ is associated with a residual wavelet which begins with two opposite peaks and gradually decays. The reason for this is simply

that an allpass distortion remains in $e_b(n)$ because only the amplitude response of nonminimum-phase source wavelet can be equalized by $v_b(n)$. The deconvolved signal $e(n)$ (dotted line) obtained by the optimum inverse filter depicted by a dash-dotted line in Figure 2(a) is shown in Figure 2(c) for $N = 0 \sim 511$ together with the true input sequence $u(n)$ (solid line). One can see, from Figure 2(c), that $e(n)$ approximates $u(n)$ well except for a scale factor. Comparing the deconvolved signal shown in Figure 2(b) with the one shown in Figure 2(c), one can easily see that $e(n)$ is indeed a much better estimate of $u(n)$ than $e_b(n)$ because the phase distortion (allpass distortion) in $e_b(n)$ (dotted line in Figure 2(b)) was almost inexistent in $e(n)$ (dotted line in Figure 2(c)). These simulation results support that the proposed inverse filter criterion $J_{2,6}$ can be used for deconvolution.

As discussed in Example 1, one surely can use Tugnait's criterion $J_{2,4}$ rather than the new criterion $J_{2,6}$ for this example because $\gamma_4 = 0.27 \neq 0$. This example only emphasizes the application of the new $J_{2,6}$ to deconvolution although both of $J_{2,4}$ and $J_{2,6}$ are members of the unified class of $J_{2s,l+s}$ where $l > s \geq 1$ presented in Theorem 1.

4. CONCLUSIONS

We have shown that the cumulant based inverse filter criteria $J_{r,m}$ given by (3) which use an mth-order cumulant and an rth-order cumulant for blind deconvolution and equalization require r to be even and $m > r$ (see Theorem 1). Therefore, these criteria form a family of criteria $J_{2s,l+s}$ where $l > s \geq 1$ and they include not only the existing inverse filter criteria as special cases of (l, s) but also new inverse filter criteria (see **(R1)**). The optimum inverse filter associated with $J_{2s,l+s}$ can only be obtained by iterative nonlinear optimization algorithms which can only guarantee a local optimum solution. Some simulation results were provided to support that $J_{2s,l+s}$ with $l > s \geq 1$ are effective.

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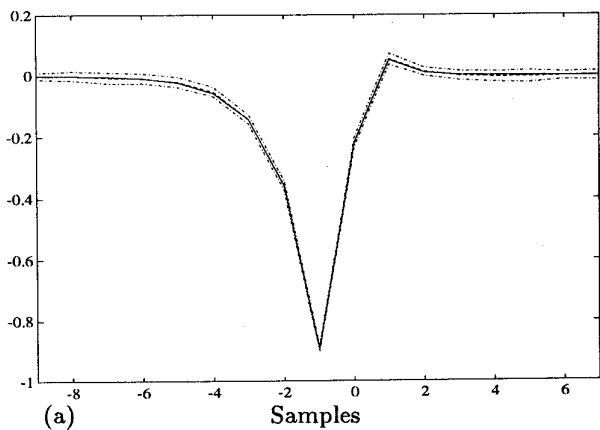[8] S. M. Kay, *Modern Spectral Estimation*, Prentice-Hall, Englewood Cliffs, New Jersey, 1988.
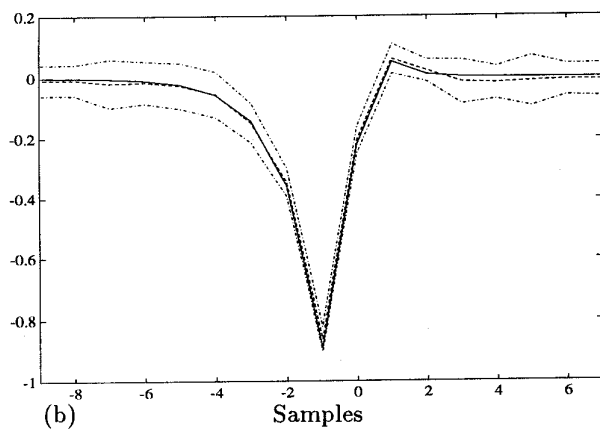
Figure 1. Simulation results ($N = 4096$, $SNR = 10dB$) for Example 1. The average (dashed line) as well as ±one standard deviation (dash-dotted lines) of 30 independent inverse filter estimates together with the true inverse filter (solid line) associated with (a) $J_{2,3}$ and (b) $J_{2,5}$, respectively.
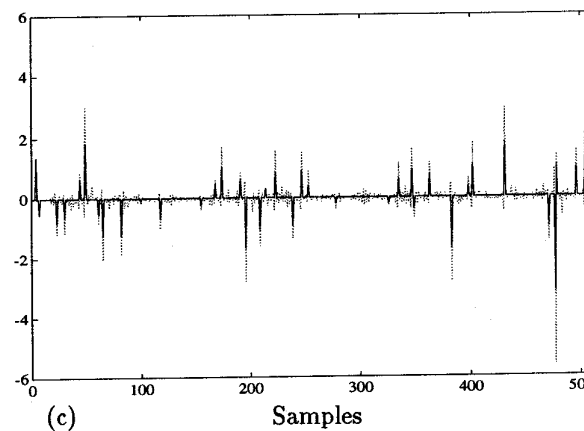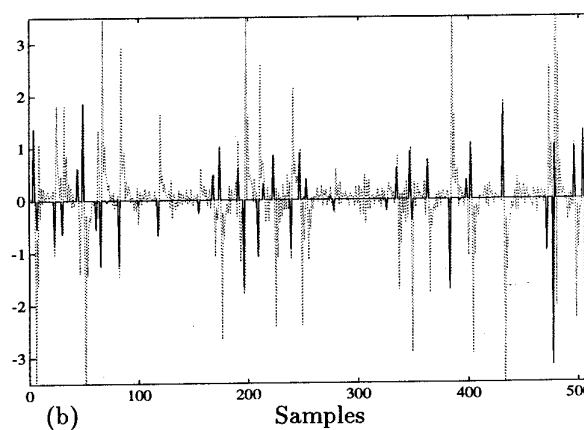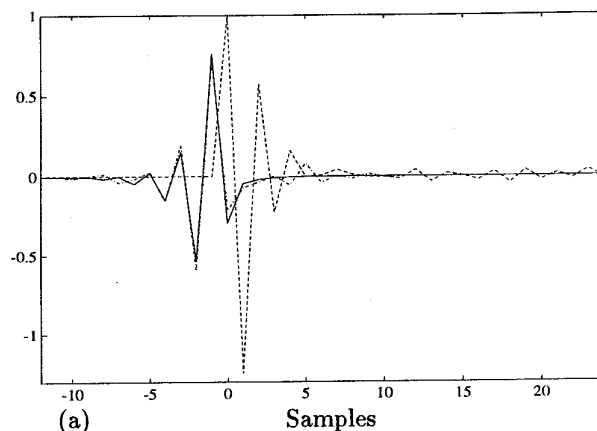

Figure 2. Simulation results ($SNR = 27dB$) associated with $J_{2,6}$ for Example 2. (a) The true noncausal stable inverse filter $h_I(n)$ (solid line), the inverse filter estimate (dash-dotted line) and the LPE filter (dashed line) of order equal to 24 obtained by Burg's algorithm; (b) the predictive deconvolved signal $e_b(n)$ (dotted line) together with the true input signal $u(n)$ (solid line); (c) the deconvolved signal $e(n)$ (dotted line) obtained by the optimum inverse filter together with the true input signal $u(n)$.